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A convergence study for the Laguerre expansion in the moment equation method for neoclassical transport in general toroidal plasmas

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The dependence of neoclassical parallel flow calculations on the maximum order of Laguerre polynomial expansions is investigated in a magnetic configuration of the Large Helical Device [S. Murakami, A. Wakasa, H. Maaßberg, *et al.*, Nucl. Fusion **42**, L19 (2002)] using the monoenergetic coefficient database obtained by an international collaboration. On the basis of a previous generalization (the so-called Sugama–Nishimura method [H. Sugama and S. Nishimura, Phys. Plasmas **15**, 042502 (2008)]) to an arbitrary order of the expansion, the 13 M, 21 M, and 29 M approximations are compared. In a previous comparison, only the ion distribution function in the banana collisionality regime of single-ion-species plasmas in tokamak configurations was investigated. In this paper, the dependence of the problems including electrons and impurities in the general collisionality regime in an actual nonsymmetric toroidal configuration is reported. In particular, qualities of approximations for the electron distribution function are investigated in detail. © *2010 American Institute of Physics*. [doi:10.1063/1.3475792]

I. INTRODUCTION

In the so-called moment equation approach to the neoclassical transport,[1–5] the Laguerre (Sonine) polynomial expansions for the Legendre order $l=1$ in distribution functions $f_{a1}^{(l=1)}$ and the linearized collision terms $C_a^{\mathrm{L}}(f_{a1}^{(l=1)})$ are used for handling the energy space.[6] (Exactly speaking, this technique is sometimes used not only for the order $l=1$ but also for $l=0$.[1] In this study investigating the flux-surface averaged flow moments in $\langle Bf_{a1}^{(l=1)}\rangle$, however, the treatment of the order $l=0$ is out of scope.) One main purpose of the expansions is to convert problems including the field particle portion $C_{ab}(\langle f_{a\mathrm{M}}\rangle, f_{b1}^{(l=1)})$ in the collision term into linear algebraic equations. Since this portion is an integral operator in the velocity space (v,ξ) [v: velocity, $\xi\equiv v_\parallel/v$: pitch-angle parameter] and causes coupling between particle species, its direct implementation in numerical algorithms for kinetic equations is nearly impossible. Therefore the moment equation approach separates the problem into two parts.[1] One part is the calculation of the viscosity coefficients for given magnetic field ($\mathbf{B}$) configurations, in which the field particle portion is not essential, and the other part is solving the parallel force balance equations expressed in the algebraic form for the given plasma parameters. Because inclusion of the field particle portion is required especially in calculation of parallel plasma flows and currents, various code developments to include it are now being performed, and it is considered that their benchmarking is also required.[7–11] However, we should note here that there are other important aspects of this orthogonal expansion corresponding to some physical laws of the order $l=1$,[12] and that this expansion can be extended to arbitrary orders.[13] Hereafter, the notations in Ref. 14, in which the treatment of general orders of the Laguerre expansion is discussed, are followed. One is that the particle and energy conservations $\nabla\cdot(n_a\mathbf{u}_a)=0$, $\nabla\cdot\mathbf{Q}_a=$*energy exchange* in the macroscopic MHD equilibrium are included as the orders $j=0,1$ in the expansion of the distribution with the flow moments $u_{\parallel aj}$,

$$f_{a1}^{(l=1)}=\frac{2v\xi}{v_{\mathrm{T}a}^2}\sum_{j=0}^{\infty}u_{\parallel aj}L_j^{(3/2)}\langle f_{a\mathrm{M}}\rangle,$$
$$u_{\parallel aj}\equiv\frac{c_j}{\langle n_a\rangle}\int v\xi L_j^{(3/2)}f_a\mathrm{d}^3\mathbf{v}, \tag{1}$$

and thus these conservation laws are retained even when the higher orders $j\geq 2$ are appropriately truncated. Here, $c_j\equiv 3\cdot 2^j j!/(2j+3)!!$, $L_j^{(3/2)}\equiv L_j^{(3/2)}(x_a^2)$, and $x_a^2\equiv m_a v^2/(2\langle T_a\rangle)\equiv(v/v_{\mathrm{T}a})^2$. Analogously the collisional momentum conservation

$$m_a\int\mathbf{v}C_{ab}(f_a,f_b)\mathrm{d}^3\mathbf{v}+m_b\int\mathbf{v}C_{ba}(f_b,f_a)\mathrm{d}^3\mathbf{v}=0 \tag{2}$$

for arbitrary pair of colliding particle species a,b concerns with only order $j=0$ in the expansion of the collision term with the friction moments $C_{\parallel aj}$,

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$$\sum_b [C_{ab}(f_{a1}^{(l=1)},\langle f_{bM}\rangle) + C_{ab}(\langle f_{aM}\rangle, f_{b1}^{(l=1)})]$$
$$= \frac{2v\xi}{v_{Ta}^2}\sum_{j=0}^{\infty} C_{\|aj} L_j^{(3/2)}\langle f_{aM}\rangle,$$
$$C_{\|aj} \equiv \frac{c_j}{\langle n_a\rangle}\int v\xi L_j^{(3/2)} C_a^{\mathrm{L}}(f_{a1}^{(l=1)})\mathrm{d}^3\mathbf{v}$$
$$= \frac{c_j}{\langle n_a\rangle m_a}\sum_b\sum_{k=0}^{\infty} l_{j+1,k+1}^{ab} u_{\|bk}. \quad (3)$$

Retaining this conservation is required also for consistency with the macroscopic MHD equilibrium $c\nabla\Sigma p_a = \mathbf{J}\times\mathbf{B}$, $\mathbf{J}\equiv\Sigma e_a n_a \mathbf{u}_a$. Furthermore, truncating higher orders $j\geq 2$ in Eqs. (1) and (3) does not break the self-adjoint property,

$$\int \hat{g}_a C_{ab}[\langle f_{aM}\rangle(1+\hat{f}_a),\langle f_{bM}\rangle(1+\hat{f}_b)]\mathrm{d}^3\mathbf{v}$$
$$+\int \hat{g}_b C_{ba}[\langle f_{bM}\rangle(1+\hat{f}_b),\langle f_{aM}\rangle(1+\hat{f}_a)]\mathrm{d}^3\mathbf{v}$$
$$=\int \hat{f}_a C_{ab}[\langle f_{aM}\rangle(1+\hat{g}_a),\langle f_{bM}\rangle(1+\hat{g}_b)]\mathrm{d}^3\mathbf{v}$$
$$+\int \hat{f}_b C_{ba}[\langle f_{bM}\rangle(1+\hat{g}_b),\langle f_{aM}\rangle(1+\hat{g}_a)]\mathrm{d}^3\mathbf{v}, \quad (4)$$

for $f_a=\langle f_{aM}\rangle(1+\hat{f}_a)$, $f_b=\langle f_{bM}\rangle(1+\hat{f}_b)$ of the arbitrary pair a,b. In Eq. (3), this characteristic for the Legendre order $l=1$ is expressed by the symmetric relation of the friction coefficient[1] $l_{ji}^{ba}=l_{ij}^{ab}$. It is also well known that[12] the momentum conservation equation (2) results in the intrinsic ambipolarity of the collisional diffusion in the axisymmetric, helically symmetric, and poloidally symmetric conditions of the **B**-field strength on the flux-surfaces,[5] and this self-adjoint property results in the Onsager symmetry of the transport matrix.[4] Confirming these characteristics of the transport coefficients is an important and convenient self-check for code developments.

Although many previous studies[1–5,7–10] were often done with the expansions retaining only two terms ($j=0,1$) in Eqs. (1) and (3) since it is a minimum choice to include the like-particle friction collision, these kinds of theories can be extended to include higher orders, as discussed in Ref. 13. Following this methodology, we previously generalized a theory for nonsymmetric toroidal configurations in Ref. 5 (Sugama–Nishimura method) to include arbitrary Laguerre orders in Eqs. (1) and (3).[14] However, it also should be noted that requiring much higher orders $j\gg 1$ in calculating $\langle Bf_{a1}^{(l=1)}\rangle$ is questionable from two viewpoints. One is that it is physically unnatural. This issue is caused by the linearizing approximation neglecting the nonlinear term $C_{ab}(f_{a1},f_{b1})$. When the kinetic equations are linearized by this kind of omission, arbitrary modes and/or orders (m,n,l,j) of the orthogonal expansions are allowed mathematically. However, the actual physics is nonlinear. The distribution functions f_a should simultaneously satisfy $f_a=\langle f_{aM}\rangle+f_{a1}>0$ and $\int v^{2n} f_a \mathrm{d}^3\mathbf{v}$, $\int \mathbf{v} v^{2n} f_a \mathrm{d}^3\mathbf{v}$=finite. Since the latter characteristic is more important in many practical applications, we use the orthogonally expanded expressions guaranteeing it. However, $f_a>0$ is not guaranteed in those kinds of linearized theories and/or codes. When truncating the Laguerre series at an appropriate order $j_{\max}$, inclusion of the higher Laguerre orders $j\gg 1$ prevents the smooth monotonic decrease of f_a keeping $f_a>0$ at a high energy tail $x_a\gg 1$. This is an essential difference between the Fourier–Legendre (m,n,l) expansion for (θ,ζ,ξ) space and the Laguerre (j) expansion for the v-space. The second problem is that increasing the matrix size often increases numerical errors. For these two reasons, the matrix size should be minimized as effectively as possible, especially when the number of particle species is large. A main conclusion of a previous comparison of so-called 13M ($j\leq j_{\max}=1$), 21M ($j\leq j_{\max}=2$), and 29M ($j\leq j_{\max}=3$) approximations to answer to this question was that the 13 M approximation is sufficiently accurate for practical purposes.[14] However, this previous numerical example in Ref. 14 was made (1) only for the ion distribution function in plasmas consisting of a single species of ion, (2) only for banana collisionality regime in tokamak configurations, and (3) by using a small mass ratio approximation to neglect the nondiagonal coupling between electrons and the ions. Therefore, the dependence of the viscosity coefficients on ambipolar radial electric fields, which is peculiar to nonsymmetric stellarator/heliotron configurations, was not included. Furthermore, the Onsager symmetric bootstrap current and Ware pinch coefficients in multi-ion-species plasmas were also not investigated in that study. In this study, the calculation is extended to fully include the nondiagonal coupling between arbitrary particle species including the electrons and impurity ions and the dependence of the flows and the transport coefficients on the $\mathbf{E}\times\mathbf{B}$ effect is investigated.

II. EXTENSION FOR THE CONVERGENCE STUDY

Previous studies often show approximated analytical solutions obtained by omitting nondiagonal coupling between particle species. An example is the 13 M approximation for plasmas consisting of only electrons and a single species of ions in Appendix C of Ref. 5, where the ion-electron collision term $C_{\mathrm{ie}}(f_{\mathrm{i}},f_{\mathrm{e}})$ and the inductive parallel electric field $\langle BE_{\|}^{(\mathbf{A})}\rangle$ in the ion moment equation are treated as the higher order of electron-ion mass ratio. In the present study investigating the bootstrap current and Ware pinch coefficients in multi-ion-species plasmas with an extension to arbitrary Laguerre orders, a method is required to fully include the coupling caused by the field particle portion. Also, the inductive parallel electric field cannot be omitted from the ion moment equations for confirming the Onsager symmetry of the bootstrap current and the Ware pinch. On this Onsager symmetry due to the self-adjoint property, Eq. (4), it was indicated in Ref. 14 that the Laguerre expansion should be performed directly for the kinetic equation. If the kinetic equation is multiplied by inappropriate weighting functions

in the v-space, the intrinsic ambipolarity and the Onsager symmetry are broken. In order to retain momentum conservation and self-adjointness, the following algebraic equation is used, which is obtained by taking the $\int vL_j^{(3/2)}\mathrm{d}^3\mathbf{v}$ Laguerre moments of orders $0 \le j \le j_{\max}$ of $\langle B\int \xi \mathrm{d}\xi\rangle$ of the drift kinetic equation itself.[14]

$$\left(-\begin{bmatrix} \mathbf{M}_a & 0 & 0 & \cdots & 0 \\ 0 & \mathbf{M}_b & 0 & \cdots & 0 \\ 0 & 0 & \mathbf{M}_c & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \mathbf{M}_N \end{bmatrix} + \begin{bmatrix} \mathbf{\Lambda}^{aa} & \mathbf{\Lambda}^{ab} & \mathbf{\Lambda}^{ac} & \cdots & \mathbf{\Lambda}^{aN} \\ \mathbf{\Lambda}^{ba} & \mathbf{\Lambda}^{bb} & \mathbf{\Lambda}^{bc} & \cdots & \mathbf{\Lambda}^{bN} \\ \mathbf{\Lambda}^{ca} & \mathbf{\Lambda}^{cb} & \mathbf{\Lambda}^{cc} & \cdots & \mathbf{\Lambda}^{cN} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \mathbf{\Lambda}^{Na} & \mathbf{\Lambda}^{Nb} & \mathbf{\Lambda}^{Nc} & \cdots & \mathbf{\Lambda}^{NN} \end{bmatrix}\right)\begin{bmatrix} \mathbf{U}_a \\ \mathbf{U}_b \\ \mathbf{U}_c \\ \cdots \\ \cdots \\ \mathbf{U}_N \end{bmatrix} = \begin{bmatrix} \mathbf{N}_a & 0 & 0 & \cdots & 0 \\ 0 & \mathbf{N}_b & 0 & \cdots & 0 \\ 0 & 0 & \mathbf{N}_c & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \mathbf{N}_N \end{bmatrix}\begin{bmatrix} \mathbf{X}_a \\ \mathbf{X}_b \\ \mathbf{X}_c \\ \cdots \\ \cdots \\ \mathbf{X}_N \end{bmatrix} - \begin{bmatrix} \mathbf{Z}_a \\ \mathbf{Z}_b \\ \mathbf{Z}_c \\ \cdots \\ \cdots \\ \mathbf{Z}_N \end{bmatrix}\langle BE_{\parallel}^{(\mathbf{A})}\rangle. \tag{5}$$

Here, $\mathbf{M}_a$ and $\mathbf{\Lambda}^{ab}$ are $(j_{\max}+1)\times(j_{\max}+1)$ matrices that include the parallel viscosity coefficients $M^a_{j+1,k+1}/\langle B^2\rangle$ expressing the damping force against the flows and the friction coefficients $l^{ab}_{j+1,k+1}$, respectively. The coupling effects between particle species are included by the nondiagonal elements $l^{ab}_{j+1,k+1}$ $(a\neq b)$ corresponding to the field particle portion $C_{ab}(\langle f_{a\mathrm{M}}\rangle, f_{b1}^{(l=1)})$. The driving forces for the flux-surface-averaged flow moments $\mathbf{U}_a = [\langle Bu_{\parallel a0}\rangle, \langle Bu_{\parallel a1}\rangle, \cdots \langle Bu_{\parallel a,j_{\max}}\rangle]^{\mathrm{tr}}$ due to the radial gradient forces $\mathbf{X}_a = [X_{a1}, -X_{a2}]^{\mathrm{tr}}$ are described by $\mathbf{N}_a$, which is the $2\times(j_{\max}+1)$ matrix of the nondiagonal viscosity coefficients $N^a_{j+1,k+1}$. To include the inductive field $\langle BE_{\parallel}^{(\mathbf{A})}\rangle$, $\mathbf{Z}_a = [e_a\langle n_a\rangle, 0, 0, \ldots, 0]^{\mathrm{tr}}$, which are vectors with the size of $(j_{\max}+1)$, are used. The friction coefficients l^{ab}_{ij} for arbitrary Laguerre orders and for arbitrary pair of colliding species a,b can be derived by a method that uses the generating function for Sonine polynomials.[6] It should also be noted that the higher orders with respect to the electron-ion mass ratio $m_{\mathrm{e}}/m_{\mathrm{i}}$ should be omitted to retain the symmetric relation $l^{ba}_{ji} = l^{ab}_{ij}$ in cases of their different temperatures $T_{\mathrm{e}} \neq T_{\mathrm{i}}$, and also terms of $(m_{\mathrm{e}}/m_{\mathrm{i}})^{1/2}$ in the ion $(a\neq \mathrm{e})$ friction moment should be retained for the momentum conservation.

By solving this algebraic equation, $\mathbf{U}_a$ are given as the linear combination of $\mathbf{X}_a$ and $\langle BE_{\parallel}^{(\mathbf{A})}\rangle$. We use the direct numerical solution of this equation including the full matrix elements. Thus, the transport coefficients (diffusions, Ware pinch, bootstrap current, and parallel conductivity) are now defined by this numerical solution and cannot be expressed analytically. Though this fact may seem complicated, for computers, it is a simple and straightforward calculation.

III. VISCOSITY COEFFICIENTS

As shown in Ref. 14, the viscosity coefficients $M^a_{j+1,k+1}$ and $N^a_{j+1,k+1}$ in the previous section are obtained by energy integrals of the monoenergetic coefficients $M_a(K)$ and $N_a(K)$ as functions of $K \equiv x_a^2$ given by solving the monoenergetic differential equations $(V_{\parallel} + V_E - C_a^{\mathrm{PAS}})G_{Xa} = \sigma_{Xa}$ and $(V_{\parallel} + V_E - C_a^{\mathrm{PAS}})G_{Ua} = \sigma_{Ua}$ which describe the configuration dependence. Here, $V_{\parallel} \equiv v_{\parallel}\mathbf{b}\cdot\nabla_{(\mu=\mathrm{const})} = v\xi\mathbf{b}\cdot\nabla_{(\xi=\mathrm{const})} - (v/2)\times(\mathbf{b}\cdot\nabla \ln B)(1-\xi^2)\partial/\partial\xi$, where $\mathbf{b} = \mathbf{B}/B$, is the parallel orbit propagator, $V_E \equiv cE_s(\nabla s \times \mathbf{B}/\langle B^2\rangle)\cdot\nabla_{(\xi=\mathrm{const})}$ is the $\mathbf{E}\times\mathbf{B}$ operator describing an effect of the ambipolar radial electric field $E_s \equiv -\mathrm{d}\Phi/\mathrm{d}s$ on the guiding center orbits [s: arbitrary label of the flux-surfaces], and $C_a^{\mathrm{PAS}} \equiv (\nu_{\mathrm{D}}^a/2)(\partial/\partial\xi)(1-\xi^2)\times(\partial/\partial\xi)$ is the pitch-angle-scattering (PAS) collision operator. The source term σ_{Xa} in the former equation corresponds to the radial guiding center drift discussed later, and that in the latter equation, which describes the contribution of parallel flows on the viscosity, is defined as $\sigma_{Ua} \equiv -m_a(V_{\parallel} + V_E)(v\xi B)$. Since one purpose of this study is the interinstitute benchmarking of the codes having analogous functions,[7–11] this method to solve the equations should be commonly used ones. Therefore, we basically use the drift kinetic equation solver (DKES) (Ref. 15) monoenergetic coefficient database obtained in an international collaboration for benchmarking these kinds of monoenergetic coefficients.[16] The assumed magnetic configuration is that with the vacuum axis position of $R_{\mathrm{ax}} = 3.6$ m in the large helical device (LHD).[17]

For the parameter (n_{e}, T_{e}, T_{i}, Z_{eff}, and E_s) range scan with multiple-species of ions, however, this energy integral procedure requires a wide range space of the collisionality and the $\mathbf{E}\times\mathbf{B}$ parameter $(\nu/v, E_s/v)$ of the monoenergetic coefficients. This section describes methods to extend the $(\nu/v, E_s/v)$ range for the present study. The function G_{Xa} in Refs. 5 and 14 is defined by assuming that the Legendre order $l=0$ of the distribution $f_{a1}^{(l=0)}$ is determined by the momentum conserving collision operator satisfying Eq. (2) and thus $V_E f_{a1}^{(l=0)}$ is negligibly small. In Ref. 5, the relation between this function and another function F_1 calculated in the DKES (Ref. 15) was explained by neglecting not only $V_E f_{a1}^{(l=0)}$ but also $V_E F_1^{(l=0)}$. However, the assumption of $V_E F_1^{(l=0)} \simeq 0$ is often violated[7] since an inappropriate collision operator that breaks the momentum conservation determines this function. We analyze this problem first. The equation of G_{Xa} given by

$$(V_\parallel + V_E - C_a^{\mathrm{PAS}})G_{Xa} = \sigma_{Xa} \equiv -\sigma_1^+ - \frac{m_a c}{e_a}(V_\parallel + V_E)(v\xi\tilde{U}) \tag{6}$$

in Refs. 5 and 14 can be rewritten as

$$(V_\parallel + V_E - C_a^{\mathrm{PAS}})\left(G_{Xa} + \frac{m_a c}{e_a}v\xi\tilde{U}\right) = -\sigma_1^+ + \nu_{\mathrm{D}}^a \frac{m_a c}{e_a}v\xi\tilde{U}. \tag{7}$$

Here, $\sigma_1^+ \equiv -\mathbf{v}_{\mathrm{d}a}\cdot\nabla s \equiv (c/e_a)(m_a v^2/4)(1+\xi^2)(\nabla s\times\mathbf{B})\cdot\nabla(1/B^2)$ is the full component of the radial drift velocity, and $\tilde{U}$ is the solution of $\mathbf{B}\cdot\nabla(\tilde{U}/B) = (\mathbf{B}\times\nabla s)\cdot\nabla(1/B^2)$ described in Appendix A of Ref. 5. On the other hand,

$$(V_\parallel + V_E - C_a^{\mathrm{PAS}})F_1 = \sigma_1^+, \tag{8}$$

which is solved in the DKES,[15] is equivalent to the following equation:

$$(V_\parallel + V_E - C_a^{\mathrm{PAS}})\left(F_1 - \nu_{\mathrm{D}}^a \frac{m_a c}{e_a}\int^l \tilde{U}\mathrm{d}l\right) = \sigma_1^+ - \nu_{\mathrm{D}}^a \frac{m_a c}{e_a}v\xi\tilde{U} - \nu_{\mathrm{D}}^a \frac{m_a c}{e_a}V_E\int^l \tilde{U}\mathrm{d}l. \tag{9}$$

When the last term in Eq. (9) is negligible, by comparing Eqs. (7) and (9), we obtain

$$G_{Xa} + \frac{m_a c}{e_a}v\xi\tilde{U} \cong -F_1 + \nu_{\mathrm{D}}^a \frac{m_a c}{e_a}\int^l \tilde{U}\mathrm{d}l. \tag{10}$$

Although it is suggested by Eq. (42) in Ref. 5 that we can obtain G_{Xa} by solving Eq. (8) (DKES) instead of Eq. (6) with this relation, this substitution is justified only in cases of $|(e_a/c)\sigma_1^+| \gg m_a\nu_{\mathrm{D}}^a|V_E\int^l\tilde{U}\mathrm{d}l|$. The violation of the validity of Eq. (8) in the large $\nu_{\mathrm{D}}^a E_s$ limit appearing especially for impurity ions in high density and/or high-Z_{eff} conditions is caused by violation of the momentum conservation. When Eq. (8) is not valid, we cannot use D_{11} and $D_{13}=D_{31}$ defined in Ref. 15 for calculating the neoclassical viscosity effects. The second problem preventing our straightforward energy integral procedure including the large ν/v limits may be the energy scattering correction to the collision of the Legendre order $l=2$. An essential consideration stated in Refs. 1, 5, and 14 on this correction is that the Pfirsch–Schlüter (PS) energy range, in which $C_a^{\mathrm{PAS}} f_{a1}^{(l=2)} = -3\nu_{\mathrm{D}}^a f_{a1}^{(l=2)}$ should be replace by the anisotropy relaxation Krook term $-\nu_{\mathrm{T}}^a f_{a1}^{(l=2)}$, is defined by $\nu_{\mathrm{T}}^a/v > 1/L_c$, where L_c is the characteristic length of the B-field modulation $1/L_c \simeq \mathbf{b}\cdot\nabla\ln B$. However, this definition is not clear and straightforward in stellarator/heliotron configurations, since two types of modulation coexist: (1) high frequency modulation due to the ripple structure and (2) low frequency modulation due to the toroidicity. A quantitatively clear definition for this situation is $\nu_{\mathrm{T}}^a/v > (8/5\pi)\times(4\pi^2/V')|\chi' m - \psi' n|/\langle B^2\rangle^{1/2}$ for individual Fourier modes $\sin(m\theta - n\zeta)$, $\cos(m\theta - n\zeta)$ of the distribution and the B-field modulation. Here, χ', ψ', and V' are radial derivatives $\mathrm{d}/\mathrm{d}s$ of the poloidal flux, toroidal flux, and the volume enclosed by the flux-surface $s=$const, respectively. In this study requiring the extension to the large ν/v limit with the finite E_s/v values, we shall use the analytical formulas for the plateau and PS energy range shown in Ref. 18. Since these formulas are obtained by solving equations of G_{Xa} and G_{Ua} directly, the physically meaningless contribution of the $V_E F_1^{(l=0)}$ is excluded. The dependence of the energy scattering correction on the characteristic length L_c is naturally and automatically included by the Fourier expansion. Though the method to include the resonant viscosity effect[19] is not shown in that study, we can include it by following connection of the monoenergetic analytical formulas in E_s/v space. The plateau regime monoenergetic viscosity coefficients that include this effect in the large E_s/v limit can be obtained by a method using the aforementioned Krook collision term $-\nu_{\mathrm{T}}^a$ and the Fourier expansion.[19] Though this approximated solution with $E_s/v=0$ in collisionless and collisional limits of $\nu/v\to 0,\infty$ reproduces the exact plateau and PS asymptotic values, as a result of a break of collisional particle and energy conservation $\int\mathrm{d}\xi C_a^{\mathrm{L}}(G_{Xa}\langle f_{a\mathrm{M}}\rangle) = \int\mathrm{d}\xi C_a^{\mathrm{L}}(G_{Ua}\langle f_{a\mathrm{M}}\rangle) = 0$, the approximation becomes worth at the transition condition $\nu_{\mathrm{T}}^a/v \simeq (8/5\pi)(4\pi^2/V')|\chi' m - \psi' n|/\langle B^2\rangle^{1/2}$ of each Fourier modes (m,n). Its use should be limited to the large E_s/v range of $|b_{mn}| > 3.0\times 10^{-2}$, where $b_{mn} \equiv (cE_s/v)\times\langle B^2\rangle^{-1/2}(B_\zeta m + B_\theta n)/(\chi' m - \psi' n)$. Therefore the monoenergetic viscosity coefficients in the plateau and PS energy ranges are given by connecting the function

$$\left\{\left(\frac{8}{\pi}|\chi' m - \psi' n|\right)^{3/2} + \left(5\frac{\nu_{\mathrm{T}}^a}{v}\langle B^2\rangle^{1/2}\frac{V'}{4\pi^2}\right)^{3/2}\right\}^{-2/3} \tag{11}$$

in Ref. 18 for $|b_{mn}| < 3.0\times 10^{-2}$ to the other function expressing the resonance effect,

$$\frac{9a_{mn}}{8|\chi' m - \psi' n|}\int_{-1}^{1}\frac{(\xi^2 - 1/3)^2}{(\xi - b_{mn})^2 + a_{mn}^2}\mathrm{d}\xi, \qquad a_{mn} \equiv \frac{\nu_{\mathrm{T}}^a}{v}\frac{\langle B^2\rangle^{1/2}}{|\chi' m - \psi' n|}\frac{V'}{4\pi^2}, \tag{12}$$

at $|b_{mn}| = 3.0\times 10^{-2}$ in the E_s/v space. Since this pitch-angle integral can be executed analytically, inclusion of this resonance viscosity effect in the energy integrated coefficients $M_{j+1,k+1}^a$, $N_{j+1,k+1}^a$, and L_j^a does not require a large increase in the computational time.

Figure 1 shows examples of the monoenergetic coefficients (for electrons with $\nu_{\mathrm{T}}^a \cong 3\nu_{\mathrm{D}}^a$) obtained by the connection in the v-space. The open and closed symbols indicate conditions with relatively smaller and larger $\mathbf{E}\times\mathbf{B}$ parameters E_r/v, respectively. One distinctive characteristic of nonsymmetric heliotron/stellarator configurations can be seen in Fig. 1(b). Compared with symmetric configurations, a deviation from $G^{(\mathrm{BS})} = $const in the v-space may lead to a

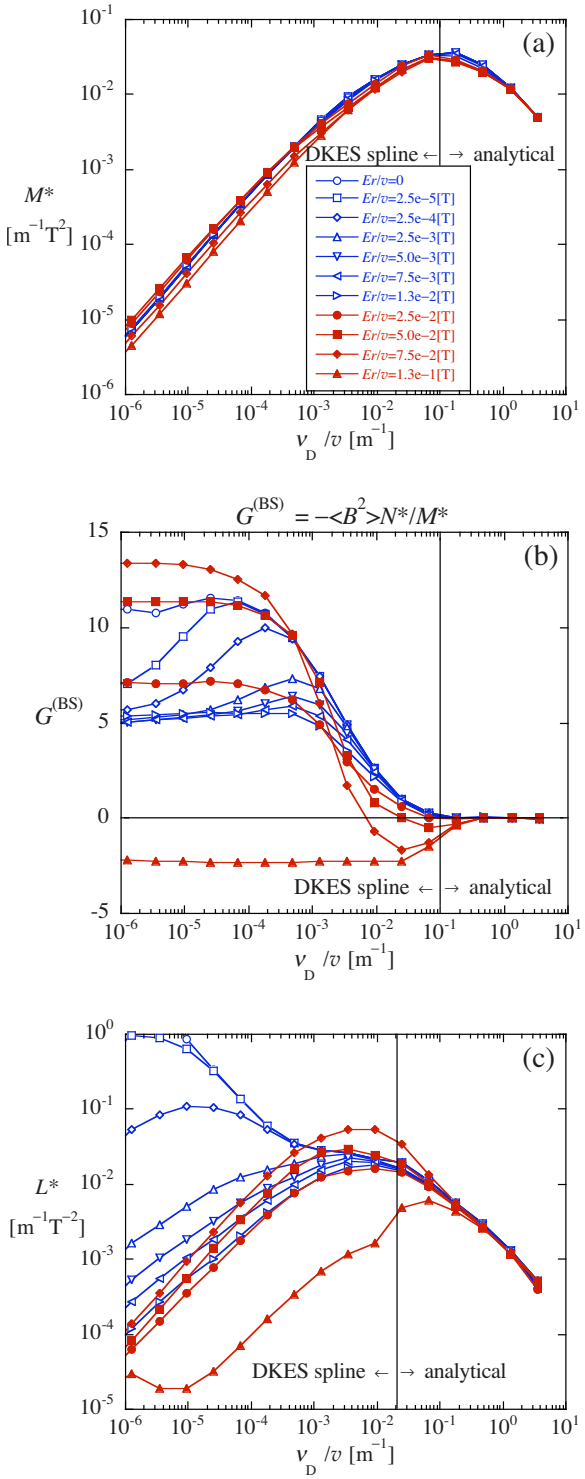

FIG. 1. (Color online) Normalized monoenergetic viscosity coefficients M^*, N^*, and L^*, which are defined in Refs. 5, 14, and 18, at a radial position of $r/a \cong 0.5$ in the LHD configuration ($R_{ax}=3.6$ m, $B_{00}=2.45$ T). Hereafter, the label of the flux-surfaces is the surface averaged minor radius ($s=r$ [m]) as in these references. The nondiagonal coefficient N^* is shown in a normalized form $G^{(BS)} \equiv -\langle B^2\rangle N^*/M^*$ (so-called geometrical factor) (Refs. 2–5).

complicated fine structure of $\langle Bf_{a1}^{(l=1)}\rangle$ in the v-space, especially when the finite $\mathbf{E}\times\mathbf{B}$ effects are included. One effect is the ripple-trapped/untrapped boundary layer effect,[18] which is suppressed by a weaker E_r/v. Another is the resonance viscosity at a large E_r/v. Even when $cE_r/B \ll v_{Ti}$, which is required for consistency with the stellartor/heliotron equilibrium[20] with $n_a m_a \mathbf{u}_a \cdot \nabla \mathbf{u}_a \ll \nabla p_a \Leftrightarrow u_a \ll v_{Ta}$, is satisfied, the first toroidal resonant condition $cE_r/B \approx v_{Ti}(\iota/2\pi) \times \langle r\rangle/\langle R\rangle$ for the Fourier mode $(m,n)=(1,0)$ often occurs for collisional impurity ions in inner regions in heliotron configurations.[21] Although we focus on the approximation method for the flux-surface-averaged flow moments in $\langle Bf_{a1}^{(l=1)}\rangle$ and thus components of radial diffusion fluxes caused by the diagonal coefficient L^* are basically out of scope of this paper, also show a connecting result of this coefficient in Fig. 1(c). This result indicates that Eq. (12) can approximate the resonance of the mode $(m,n)=(1,0)$ obtained by the DKES with the pitch-angle-scattering operator that does not break the collisional particle and energy conservation in Eq. (6) at $\nu_D^a/v \approx 0.01$ m^{-1}.

IV. RESULTS

The results of the parallel force balance equation (5) with the viscosity coefficients at the radial position $r/a \cong 0.5$ (Fig. 1) are shown in this section. In Figs. 2 and 3, open and closed circles indicate $\langle Bu_{\parallel a0}\rangle$ and $\langle Bu_{\parallel a1}\rangle$ in Eq. (1), respectively, and other symbols indicate higher order ($j>1$) moments in the 21 M and 29 M approximations. In Fig. 4, open and closed circles indicate the bootstrap current and Ware pinch coefficients $L_{E1}^a=-L_{1E}^a$ and $-L_{E2}^a=L_{2E}^a$ in the transport matrix,[5]

$$\begin{bmatrix} \langle \mathbf{\Gamma}_a^{bn}\cdot\nabla s\rangle \\ \langle \mathbf{q}_a^{bn}\cdot\nabla s\rangle/\langle T_a\rangle \end{bmatrix} = \sum_b \begin{bmatrix} L_{11}^{ab} & L_{12}^{ab} \\ L_{21}^{ab} & L_{22}^{ab} \end{bmatrix} \begin{bmatrix} X_{b1} \\ X_{b2} \end{bmatrix} + \begin{bmatrix} L_{1E}^a \\ L_{2E}^a \end{bmatrix} X_E, \tag{13}$$

$$\langle \mathbf{B}\cdot\mathbf{J}\rangle/\langle B^2\rangle^{1/2} = \sum_a \begin{bmatrix} L_{E1}^a & L_{E2}^a \end{bmatrix} \begin{bmatrix} X_{a1} \\ X_{a2} \end{bmatrix} + (L_{EE}+\sigma_S)X_E,$$

with the thermodynamic forces defined by

$$X_{a1} \equiv -\frac{1}{\langle n_a\rangle}\frac{\partial\langle p_a\rangle}{\partial s} - e_a\frac{\partial\langle\Phi\rangle}{\partial s}, \quad X_{a2} \equiv -\frac{\partial\langle T_a\rangle}{\partial s}, \tag{14}$$

$$X_E \equiv \langle BE_\parallel^{(\mathbf{A})}\rangle/\langle B^2\rangle^{1/2}.$$

In all of these results, thick solid, dot, and thin solid curves indicate the 13 M, 21 M, and 29 M approximations, respectively. As mentioned in the previous sections, these examples take into account two types of energy scattering collisions. One type is that for the Legendre order $l=1$ of the collision term $C_a^L(f_{a1}^{(l=1)})$ expressed by the friction coefficient $l_{j+1,k+1}^{ab}$ in Eqs. (3) and (5). It reduces the higher Laguerre order coefficients $\langle Bu_{\parallel aj}\rangle$ in Eq. (1). The second is that for the order $l=2$ included in the anisotropy relaxation rate ν_T^a in Eqs. (11) and (12), which reduces the higher order coefficients $M_{j+1,k+1}^a$, $N_{j+1,k+1}^a$ with $j\geq 2$ in Eq. (5). These effects are large for heavy and high-Z species and are dominated over by the pitch-angle-scattering in collision terms of light and low-Z species. This difference can be seen in the results. Though the result in Fig. 2 for plasmas with single species of the ion with $Z_{eff}=1$ indicates that the bootstrap current coefficients L_{E1}^a and L_{E2}^a, which are determined only by

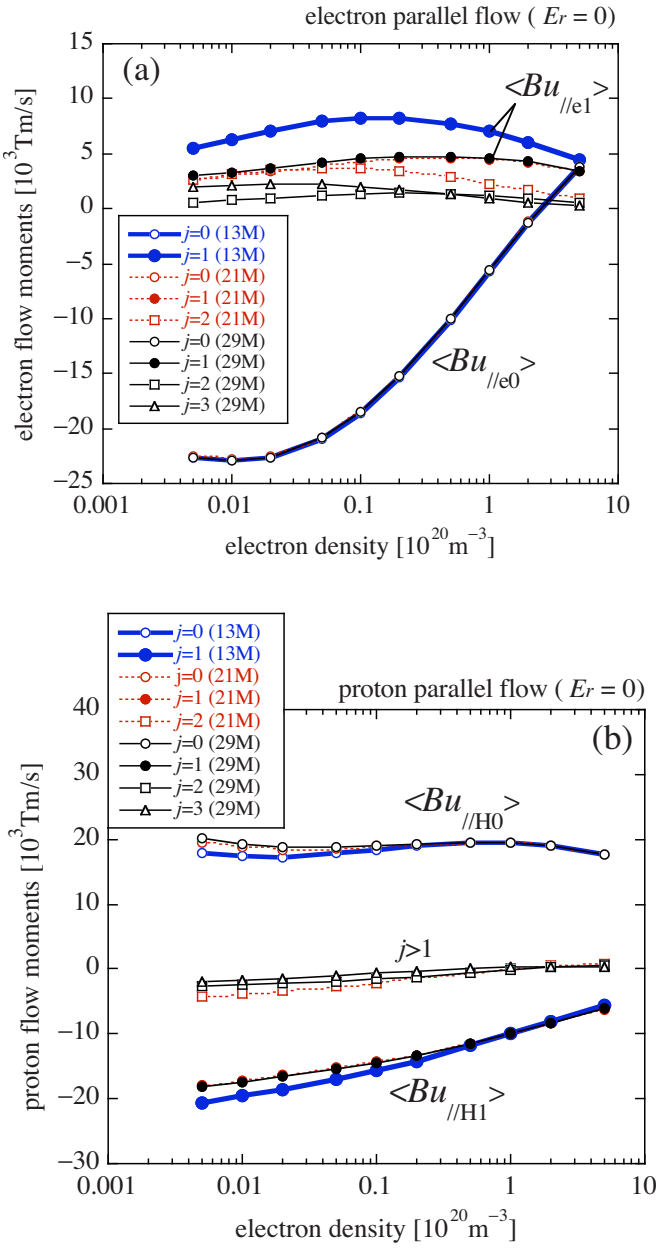

FIG. 2. (Color online) A comparison of flow moments $\langle Bu_{\|aj}\rangle$ for (a) $a=e^-$ and (b) $a=H^+$ obtained by the 13 M, 21 M, and 29 M approximations as functions of the electron density in e^-+H^+ plasmas. Assumed parameters are $T_e=T_i=2.0$ keV, $\partial p_e/\partial r/n_e=\partial p_i/\partial r/n_i=\partial T_e/\partial r=\partial T_i/\partial r=-3.0$ keV/m, and $E_r=\langle BE_\|^{(A)}\rangle=0$.

$\langle Bn_a u_{\|a}\rangle \equiv \langle n_a\rangle\langle Bu_{\|a0}\rangle$, can be exactly obtained by the 13 M approximation ($j_{max}=1$); the other result in Fig. 4 indicates that they require the 21 M approximation ($j_{max}=2$) when heavy high-Z impurity ions are included. This requirement is caused by the characteristic of the collision for the light low-Z species. In this situation of the light low-Z species, the structures of their $\langle Bf_{a1}^{(l=1)}\rangle$ in v-space are directly affected by those of their $N_a(K)$, $M_a(K)$. However, as shown in Figs. 3 and 4, the 29 M approximation ($j_{max}=3$) is not required for all of these situations even when the complicated E_s/v dependence of the boundary layer effect (at $E_r<2$ kV/m for H^+) the resonant viscosity effect ($E_r>5$ kV/m for Ne^{10+}) are included in $N_a(K)$. In these figures, a nonlinear dependence on the radial electric field E_r seen in $\langle Bu_{\|a0}\rangle$ of Ne^{10+}

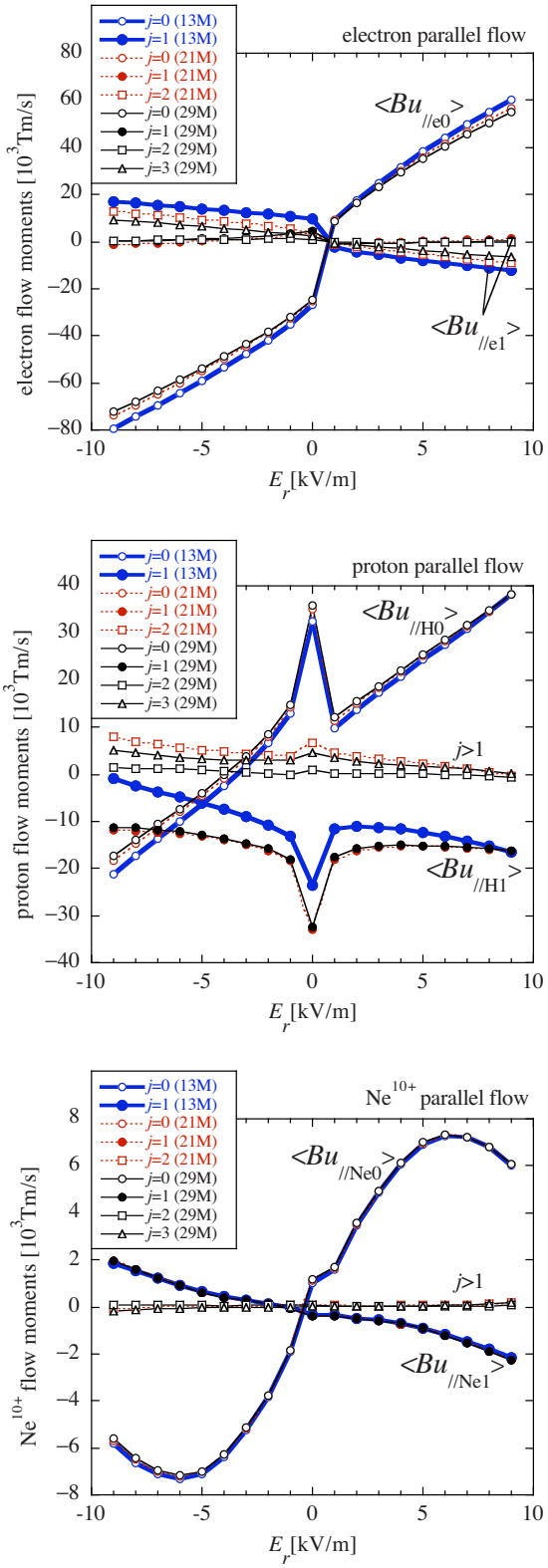

FIG. 3. (Color online) Another comparison of the flow moments $\langle Bu_{\|aj}\rangle$ for $a=e^-$, H^+, and Ne^{10+} as functions of the radial electric field strength E_r by the three approximation methods in $e^-+H^++Ne^{10+}$ plasmas. Assumed parameters are $T_e=2.0$ keV, $T_i=1.0$ keV, $Z_{eff}=5.74$, $n_e=1\times10^{18}$ m^{-3}, and $\partial p_e/\partial r/n_e=\partial p_i/\partial r/n_i=\partial T_e/\partial r=\partial T_i/\partial r=-3.0$ keV/m.

ions, whose ν/v value at the thermal velocity corresponds to the plateau regime, indicates the first toroidal resonance. Another fact, which is commonly seen in these dependences on the particle mass, collisionality, and E_r, is that $\langle Bf_{a1}^{(l=1)}\rangle$ of the

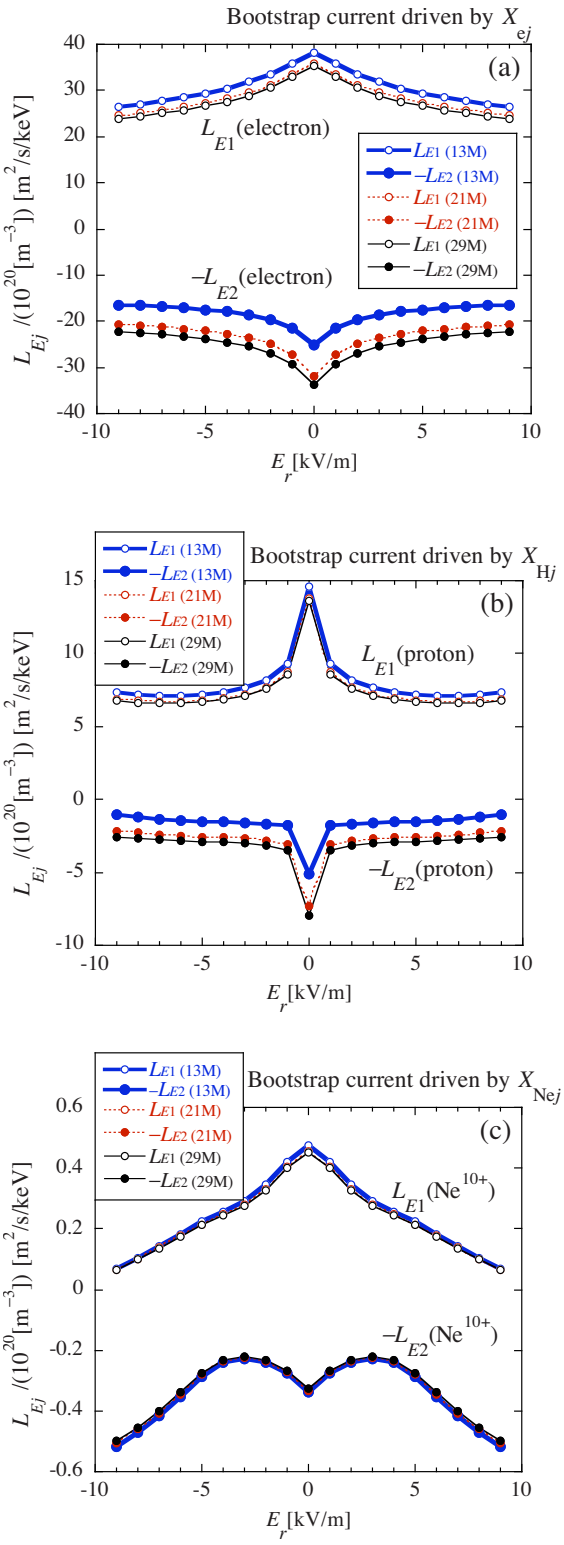

FIG. 4. (Color online) A comparison of the bootstrap current and Ware pinch coefficients $L^a_{E1}=-L^a_{1E}$ and $-L^a_{E2}=L^a_{2E}$ defined in Eq. (13) for the $e^-+H^++Ne^{10+}$ plasmas. Figures indicate the bootstrap current driven by (a) the electron forces $a=e^-$, (b) the proton forces $a=H^+$, and (c) the neon forces $a=Ne^{10+}$. The assumed parameters are those in Fig. 3.

heaviest particle species is sufficiently approximated by the expansion with only $j_{\max}=1$. This result is due to the energy scattering collision reducing the higher Laguerre order components in $\langle Bu_{\parallel aj}\rangle$, $M^a_{j+1,k+1}$, and $N^a_{j+1,k+1}$ of the heavy species.

V. CONCLUSION

The 13M ($j\le j_{\max}=1$), 21M ($j\le j_{\max}=2$), and 29M ($j\le j_{\max}=3$) approximations for the neoclassical parallel flows are compared in an actual nonaxisymmetric heliotron configuration including the finite radial electric field effects. In contrast with a previous numerical example in Ref. 14, the nondiagonal coupling effects between arbitrary particle species including the electrons are taken into account in this study, and therefore the present results will be useful for future studies on bootstrap current, helium ash control, impurity transport, and so on. The results are summarized as follows.

The flux-surface-averaged component of the distribution $\langle Bf^{(l=1)}_{a1}\rangle$ of heavy particle species can be approximated by the expansion with only two terms ($j_{\max}=1$) in most of practical cases including the situation of the first toroidal resonance of impurities, because of $l=1,2$ energy scattering collisions. The fine structure of $\langle Bf^{(l=1)}_{a1}\rangle$ in the energy (v) space is smoothed out by this energy scattering effect. Expressing these flux-surface-averaged flow moments of light particle species especially that of electrons sometimes requires the expansion with $j_{\max}=2$. The reason is that $l=1,2$ energy scattering e-e collision effects are small compared with the pitch-angle-scattering e-i collision. In this situation, the structure of $\langle Bf^{(l=1)}_{e1}\rangle$ in the v-space is directly affected by that of $N_e(K)$ and $M_e(K)$. In impure plasmas, $j_{\max}=2$ is favorable for proton distribution function by the same reason. It should be emphasized here that $j_{\max}=3$ is not required for all of these cases. Though we focused on the parallel flow moments driven by the radial gradient forces in this paper, this calculation also clarified that the parallel conductivity $L_{EE}+\sigma_S$ also is insensitive to this expansion method. Note also that this conclusion on the impurities is for the flux-surface-averaged parallel flow moments in $\langle Bf^{(l=1)}_{a1}\rangle$. For the poloidal and toroidal variations $f^{(l=1)}_{a1}-\langle Bf^{(l=1)}_{a1}\rangle B/\langle B^2\rangle$ of collisional high-Z species, $j_{\max}=2$ is required to include the $l=0$ energy scattering ion-ion collision.[1]

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